

# EQUATIONAL THEORIES OF FIELDS

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**ABSTRACT.** A complete first-order theory is equational if every definable set is a Boolean combination of instances of equations, that is, of formulae such that the family of finite intersections of instances has the descending chain condition. Equationality is a strengthening of stability. In this short note, we prove that theory of proper extension of algebraically closed fields of some fixed characteristic is equational.

## 1. INTRODUCTION

Consider a sufficiently saturated model of a complete theory  $T$ . A formula  $\varphi(x; y)$  is an *equation* (for a given partition of the free variables into  $x$  and  $y$ ) if the family of finite intersections of instances  $\varphi(x, a)$  has the descending chain condition (DCC). The theory  $T$  is *equational* if every formula  $\psi(x; y)$  is equivalent modulo  $T$  to a Boolean combination of equations  $\varphi(x; y)$ .

Quantifier elimination implies directly that the theory of algebraically closed fields of some fixed characteristic is equational. Separably closed fields of positive characteristic have quantifier elimination after adding to the ring language the so-called  $\lambda$ -functions [2]. The *imperfection degree* of a separably closed field  $K$  of positive characteristic  $p$  encodes the linear dimension of  $K$  over  $K^p$ . If the imperfection degree is finite, restricting the  $\lambda$ -functions to a fixed  $p$ -basis yields again equationality.

It is unknown whether equationality holds for the theory of a separably closed field  $K$  of infinite imperfection degree, that is, when  $K$  has infinite linear dimension over the definable subfield  $K^p$ .

Another important (expansion of a) theory of fields having infinite linear dimension over a definable subfield is the theory of an algebraically closed field with a predicate for a distinguished algebraically closed proper subfield. Any two such pairs are elementarily equivalent if and only if they have the same characteristic. They are exactly the models of the theory of Poizat's *belles paires* [11] of algebraically closed fields.

Deciding whether a particular theory is equational is generally not obvious. So far, the only known *natural* example of a stable non-equational theory is the free non-abelian finitely generated group [12]. In this short paper, we will prove that the theory of belles paires of algebraically closed fields of some fixed characteristic is equational. In Section 5 we provide an alternative proof in characteristic 0, by showing that definable sets are Boolean combination of certain definable sets,

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which are Kolchin-closed in the corresponding expansion  $\text{DCF}_0$ . A similar approach appeared already in [5] using different methods.

## 2. EQUATIONS AND INDISCERNIBLE SEQUENCES

Most of the results in this section come from [10, 6, 7]. We refer the avid reader to [9] for a gentle introduction to equationality.

We work inside a sufficiently saturated model of a complete theory  $T$ . A formula  $\varphi(x; y)$ , with respect to a given partition of the free variables into  $x$  and  $y$ , is an *equation* if the family of finite intersections of instances  $\varphi(x, a)$  has the descending chain condition (DCC). The theory  $T$  is *equational* if every formula  $\psi(x; y)$  is a Boolean combination of equations  $\varphi(x; y)$ .

Typical examples of equational theories are the theory of an equivalence relation with infinite many infinite classes, completions of the theory of  $R$ -modules and algebraically closed fields. If  $\varphi(x; y)$  is an equation, then so is  $\varphi^{-1}(y; x) = \varphi(x; y)$ . Also is  $\varphi(f(x); y)$  an equation, whenever  $f$  is a  $\emptyset$ -definable map. Finite conjunctions and disjunctions of equations are again equations.

Equationality is preserved under unnamng parameters and bi-interpretability [6]. It is unknown whether equationality holds if every formula  $\varphi(x; y)$ , with  $x$  a single variable, is a boolean combination of equations.

Equationality implies stability [10]. Observe that, if  $\varphi$  is an equation, the  $\varphi$ -definition of a type  $p$  over a set of parameters  $A$  is particularly simple: The intersection

$$\bigcap_{\varphi(x; a) \in p} \varphi(x, a)$$

is a formula  $\psi(x)$  over  $A$  contained in  $p$ . It suffices to set

$$d_p \varphi(y) = \forall x (\psi(x) \rightarrow \varphi(x; y)).$$

Instances of equations coincide with Sour-closed sets. A definable set  $X$  is *Sour-closed* if the family of finite intersections of conjugates of  $X$  under automorphisms of the ambient model has the DCC. Furthermore, a formula  $\varphi(x; y)$  is an equation if and only if every instance  $\varphi(x, a)$  is Sour-closed. Sour-closed definable sets are exactly the indiscernibly closed definable sets [7, Theorem 3.16]. A definable set is *indiscernibly closed* if, whenever  $(a_i)_{i \leq \omega}$  is an indiscernible sequence such that  $a_i$  lies in  $X$  for  $i < \omega$ , then so does  $a_\omega$ .

By extending the indiscernible sequence so that it becomes a Morley sequence over an initial segment, we conclude the following:

**Remark 2.1.** In a stable complete theory  $T$ , a definable set  $\varphi(x, a)$  is indiscernibly closed if, for every elementary substructure  $M$  and every Morley sequence  $(b_i)_{i \leq \omega}$  over  $M$  such that

$$a \downarrow_M b_i \text{ with } b_i \models \varphi(x, a) \text{ for } i < \omega,$$

then  $b_\omega$  realises  $\varphi(x, a)$  as well.

## 3. TYPES AND INDEPENDENCE

In this section, we will provide a presentation of the theory  $T_P$  of proper pairs of algebraically closed fields in some fixed characteristic. Most of the results mentioned here appear in [11, 1].

Work inside a sufficiently saturated model  $(K, E)$  of  $T_P$  in the language  $\mathcal{L}_P = \mathcal{L}_{rings} \cup \{P\}$ , where  $E = P(K)$  is the proper subfield. We will use the index  $P$  in order to refer to the expansion  $T_P$ .

A subfield  $A$  of  $K$  is *tame* if  $A$  is algebraically independent from  $E$  over  $E_A$

$$A \underset{E_A}{\downarrow} E,$$

where  $E_A$  is  $E \cap A$ . Tameness was called *P-independence* in [1], but in order to avoid a possible confusion, we have decided to use a different terminology.

The  $\mathcal{L}_P$ -type of a tame subfield of  $K$  is uniquely determined by its  $\mathcal{L}_P$ -quantifier-free type.

Recall that two subfields  $L_1$  and  $L_2$  of  $K$  are *linearly disjoint* over a common subfield  $F$ , denoted by

$$L_1 \underset{F}{\downarrow}^{\text{ld}} L_2,$$

if, whenever the elements  $a_1, \dots, a_n$  of  $L_1$  are linearly independent over  $F$ , then they remain so over  $L_2$ . Linear disjointness implies algebraic independence and agrees with the latter whenever the base field  $F$  is algebraically closed. Let us note that linear disjointness is a transitive relation: If  $F \subset D_2 \subset L_2$  is a subfield, denote by  $D_2 \cdot L_1$  the field generated by  $D_2$  and  $L_1$ . Then

$$L_1 \underset{F}{\downarrow}^{\text{ld}} L_2$$

if and only if

$$L_1 \underset{F}{\downarrow}^{\text{ld}} D_2 \quad \text{and} \quad D_2 \cdot L_1 \underset{D_2}{\downarrow}^{\text{ld}} L_2.$$

Expand the language  $\mathcal{L}_P$  as follows:

$$\mathcal{L}_D = \mathcal{L}_{rings} \cup \{R_n(x_1, \dots, x_n), \lambda_n^i(x; x_1, \dots, x_n)\}_{1 \leq i \leq n \in \mathbb{N}},$$

where we define the relation  $R_n$  as

$$K \models R_n(a_1, \dots, a_n) \iff a_1, \dots, a_n \text{ are } E\text{-linearly independent,}$$

and the  $\lambda$ -functions take values in  $E$  and are defined by the equation

$$a = \sum_{i=1}^n \lambda_n^i(a; a_1, \dots, a_n) a_i,$$

if  $K \models R_n(a_1, \dots, a_n) \wedge \neg R_{n+1}(a, a_1, \dots, a_n)$ , and are 0 otherwise. Clearly, a field  $A$  is closed under the  $\lambda$ -functions if and only if it is linearly disjoint from  $E$  over  $E_A$ . Thus, a fraction field  $K$  of an  $\mathcal{L}_D$ -substructure is automatically tame and the theory  $T_P$  has quantifier elimination [3] in the language  $\mathcal{L}_D$ .

The theory  $T_P$  is  $\omega$ -stable of Morley rank  $\omega$ . Since every subfield of  $E$  is automatically tame, the induced structure on  $E$  agrees with the field structure, so  $E$  has Morley rank 1. If  $A$  is a tame subfield, then the  $\mathcal{L}_P$ -definable closure coincides with the inseparable closure of  $A$  and its  $\mathcal{L}_P$ -algebraic closure is the field algebraic closure  $\text{acl}(A)$  of  $A$ . This implies that  $E_{\text{acl}_P(A)} = \text{acl}(E_A)$ .

**Lemma 3.1.** *Given two subfields  $A$  and  $B$  of  $K$  containing an  $\mathcal{L}_P$ -elementary substructure  $M$  of  $K$  such that  $A \downarrow_M^P B$ , then the fields  $E \cdot A$  and  $E \cdot B$  are linearly disjoint over  $E \cdot M$ .*

*Proof.* By the previous remark, it suffices to show that  $A$  and  $E \cdot B$  are linearly disjoint over  $E_A \cdot M$ . Furthermore, we may assume that  $A$  is a tame algebraically closed subfield of  $K$ , that is

$$A \downarrow_{E_A}^{\text{ld}} E.$$

Thus, let  $a_1, \dots, a_n$  in  $A$  and  $z_1, \dots, z_n$  in  $E \cdot B$ , not all zero, such that

$$\sum_{i=1}^n a_i \cdot z_i = 0.$$

We may clearly assume that all the  $z_i$ 's lie in the subring generated by  $E$  and  $B$ , so

$$z_i = \sum_{j=1}^m \zeta_j^i b_j,$$

for some  $\zeta_j^i$ 's in  $E$  and  $b_1, \dots, b_m$  in  $B$ , which we may assume to be linearly independent over  $E$ .

A straight-forward heir argument implies the existence of some  $\xi_j^i$ 's in  $E$ , not all zero, and  $c_1, \dots, c_m$  in  $M$  linearly independent over  $E$  such that

$$\sum_{i=1}^n a_i \sum_{j=1}^m \xi_j^i c_j = 0.$$

Since  $A$  is linearly disjoint from  $E$  over  $E_A$ , we may assume that the  $\xi_j^i$ 's lie in  $E_A$ . Since the  $\xi_j^i$ 's are not all zero, we have that, for some  $1 \leq i \leq n$ ,

$$\sum_{j=1}^m \xi_j^i c_j \neq 0,$$

as desired.  $\square$

In order to isolate certain formulae which will determine the type of a tuple in the theory of belles paires of  $\text{ACF}_0$ , we require certain basic notions from linear algebra (cf. [4, R  sultats d'Alg  bre]).

Let  $V$  be a vector subspace of  $E^n$  with basis  $\{v_1, \dots, v_k\}$ . Observe that

$$V = \{v \in E^n \mid v \wedge (v_1 \wedge \dots \wedge v_k) = 0 \text{ in } \bigwedge^{k+1} E^n\}.$$

Thus, the vector  $v_1 \wedge \dots \wedge v_k$  depends only on  $V$ , up to scalar multiplication, and determines  $V$  completely. The *Pl  ijcker coordinates*  $\text{Pk}(V)$  of  $V$  are the homogeneous coordinates of this vector with respect to the canonical basis of  $\bigwedge^k E^n$ . Let  $\text{Gr}_k(E^n)$ , the  $k^{\text{th}}$ -*Grassmannian* of  $E^n$ , denote the collection of Pl  ijcker coordinates of all  $k$ -dimensional subspaces of  $E^n$ . Clearly  $\text{Gr}_k(E^n)$  is contained in  $\mathbb{P}^{r-1}(E)$ , for  $r = \binom{n}{k}$ .

The  $k^{\text{th}}$ -Grassmannian is Zariski-closed. Indeed, given an element  $\zeta$  of  $\bigwedge^k E^n$ , there is a smallest vector subspace  $V_\zeta$  of  $E^n$  such that  $\zeta$  belongs to  $\bigwedge^k V_\zeta$ . The

vector space  $V_\zeta$  is the collection of inner products  $e \lrcorner \zeta$ , for  $e$  in  $\bigwedge^{k-1}(E^n)^*$ . Recall that the inner product is a bilinear map

$$\lrcorner : \bigwedge^{k-1}(E^n)^* \times \bigwedge^k(E^n) \rightarrow E.$$

A non-trivial element  $\zeta$  of  $\bigwedge^k E^n$  determines a  $k$ -dimensional subspace of  $E^n$  if and only if

$$\zeta \wedge (e \lrcorner \zeta) = 0,$$

for every  $e$  in  $\bigwedge^{k-1}(E^n)^*$ . Letting  $e$  run over a fixed basis of  $\bigwedge^{k-1}(E^n)^*$ , we see that the  $k^{\text{th}}$ -Grassmannian is the zero-set of a finite collection of homogeneous polynomials.

Fix some enumeration  $(M_i(x))_{i=1,2,\dots}$  of all possible monomials in  $s$  variables. Given a tuple  $a$  of length  $s$ , denote

$$\text{Ann}_n(a) = \left\{ (\lambda_1, \dots, \lambda_n) \in E^n \mid \sum_{i=1}^n \lambda_i \cdot M_i(a) = 0 \right\}.$$

**Notation.** Given tuples  $x$  and  $y$  of length  $n$ , we will denote by  $x \cdot y$  the scalar multiplication  $\sum_{i=1}^n x_i \cdot y_i$ . Therefore,

$$\text{Ann}_n(a) = \{ \lambda \in E^n \mid \lambda \cdot (M_1(a), \dots, M_n(a)) = 0 \}.$$

**Lemma 3.2.** *Two tuples  $a$  and  $b$  have the same type in  $T_P$  if and only if for every  $n$  in  $\mathbb{N}$ , we have that  $\text{ldim}_E \text{Ann}_n(a) = \text{ldim}_E \text{Ann}_n(b)$  and the type  $\text{tp}(\text{Pk}(\text{Ann}_n(a)))$  equals  $\text{tp}(\text{Pk}(\text{Ann}_n(b)))$  (in the pure field language).*

*Proof.* We need only prove the right-to-left implication. Since  $\text{Pk}(\text{Ann}_i(a))$  is determined by  $\text{Pk}(\text{Ann}_n(a))$ , for  $i \leq n$ , we obtain an automorphism of  $E$  mapping  $\text{Pk}(\text{Ann}_n(a))$  to  $\text{Pk}(\text{Ann}_n(b))$  for all  $n$ . This automorphism maps  $\text{Ann}_n(a)$  to  $\text{Ann}_n(b)$  for all  $n$  and hence extends to an isomorphism of the rings  $E[a]$  and  $E[b]$ . It clearly extends to a field isomorphism of the tame subfields  $E(a)$  and  $E(b)$ , mapping  $a$  to  $b$  and  $E$  to itself, so  $a$  and  $b$  have the same  $T_P$ -type, as required.  $\square$

**Definition 3.3.** Let  $x$  be a tuple of variables. A formula  $\varphi(x)$  in the language  $\mathcal{L}_P$  is *tame* if there are polynomials  $p_1, \dots, p_m$  in  $\mathbb{Z}[X, Z]$ , homogeneous in the variables  $Z$ , such that

$$\varphi(x) = \exists \zeta \in P^r \left( \neg \zeta \doteq 0 \wedge \bigwedge_{j \leq m} p_j(x, \zeta) \doteq 0 \right).$$

**Proposition 3.4.** *Two tuples  $a$  and  $b$  have the same type in  $T_P$  if and only if they satisfy the same tame formulae.*

*Proof.* Let  $p_1(Z), \dots, p_m(Z)$  be homogeneous polynomials over  $\mathbb{Z}$ . By Lemma 3.2, it suffices to show that

$$\ll \text{Ann}_n(a) \text{ has a } k\text{-dimensional subspace } V \text{ such that } \bigwedge_{j \leq m} p_j(\text{Pk}(V)) = 0 \gg$$

is expressible by a tame formula. Indeed, it suffices to guarantee that there is an element  $\zeta$  in  $\text{Gr}_k(E^n)$  such that

$$(e \lrcorner \zeta) \cdot (M_1(a), \dots, M_n(a)) = 0$$

for all  $e$  from a fixed basis of  $\bigwedge^{k-1}(E^n)^*$ , and

$$\bigwedge_{j \leq m} p_j(\zeta) = 0.$$

In particular, the tuple  $\zeta$  is not trivial, so we conclude that the above is a tame formula.  $\square$

By compactness, we conclude the following:

**Corollary 3.5.** *In the theory  $T_P$  of proper pairs of algebraically closed fields, every formula is a Boolean combination of tame formulae.*

Let us conclude with an observation regarding projections of certain varieties.

**Remark 3.6.** Though the theory of algebraically closed fields has elimination of quantifiers, the projection of a Zariski-closed set need not be again closed. For example, the closed set

$$V = \{(x, z) \in K \times K \mid x \cdot z = 1\}$$

projects onto the open set  $\{x \in K \mid x \neq 0\}$ . An algebraic variety  $Z$  is *complete* if, for all varieties  $X$ , the projection  $X \times Z \rightarrow X$  is a Zariski-closed map. Projective varieties are complete.

#### 4. EQUATIONALITY

In order to show that the stable theory  $T_P$  of proper pairs of algebraically closed fields is equational, it suffices to show by Corollary 3.5 that tame formulae, according to some fixed partition of the variables, are equations, or equivalently, that every instance of such formulae is indiscernibly closed.

We begin with a special case as an auxiliary result. Remember that  $x$  and  $a$  are finite tuples of variables and parameters respectively.

**Lemma 4.1.** *Let  $\varphi(x, a)$  be an instance of a tame formulae. The set*

$$\varphi(x, a) \wedge x \in P$$

*is Zariski-closed in  $E$ . In particular, the formula  $\varphi(x; y) \wedge x \in P$  is an equation.*

*Proof.* Suppose that the formula  $\varphi(x, a)$  has the form

$$\varphi(x, a) = \exists \zeta \in P^r \left( \neg \zeta \doteq 0 \wedge \bigwedge_{j \leq m} p_j(x, a, \zeta) \doteq 0 \right).$$

for some polynomials  $p_1, \dots, p_m$  with integer coefficients and homogeneous in  $\zeta$ . Express each of the monomials in  $a$  appearing in the above equation as a linear combination of a basis of  $K$  over  $E$ . We see that there are polynomials  $q_1, \dots, q_s$  with coefficients in  $E$ , homogeneous in  $\zeta$ , such that

$$\varphi(x, a) \wedge x \in P = \exists \zeta \in P^r \left( \neg \zeta \doteq 0 \wedge \bigwedge_{j \leq s} q_j(x, \zeta) \doteq 0 \right).$$

Working inside the algebraically closed subfield  $E$ , the expression inside the brackets is a projective variety, which is hence complete. By Remark 3.6, its projection is again a Zariski-closed variety, as desired.  $\square$

**Proposition 4.2.** *Tame formulae are equations.*

*Proof.* We need only show that every instance  $\varphi(x, a)$  of a tame formula is indiscernibly closed. By Remark 2.1, it suffices to consider a Morley sequence  $(b_i)_{i \leq \omega}$  over an elementary substructure  $M$  of  $(K, E)$  with

$$a \underset{M}{\downarrow}^P b_i \text{ with } \models \varphi(b_i, a) \text{ for } i < \omega.$$

In particular, the sequence  $(b_i)_{i < \omega}$  is Morley over  $M \cup \{a\}$ . By Lemma 3.1, the fields  $E(M, a)$  and  $E(M, b_i)$  are linearly disjoint over  $E(M)$  for every  $i < \omega$ . A basis  $\bar{c}$  of  $E(M, a)$  over  $E(M)$  remains thus linearly independent over  $E(M, b_i)$ .

Suppose now that the formula  $\varphi(x, a)$  has the form

$$\varphi(x, a) = \exists \zeta \in P^r \left( \neg \zeta \doteq 0 \wedge \bigwedge_{j \leq m} p_j(x, a, \zeta) \doteq 0 \right),$$

for polynomials  $p_1, \dots, p_n$  with integer coefficients and homogeneous in  $\zeta$ . By appropriately writing each monomial on  $a$  in terms of the basis  $\bar{c}$ , we have that

$$\varphi(x, a) = \exists \zeta \in P^r \left( \neg \zeta \doteq 0 \wedge \bigwedge_{j \leq s} \bar{q}_j(x, \zeta) \cdot \bar{c} \doteq 0 \right),$$

where the tuples of polynomials  $\bar{q}_j$  have now coefficients in  $E(M)$  and are homogeneous in  $\zeta$ . Hence, linearly disjointness implies that

$$\models \exists \zeta \in P^r \left( \neg \zeta \doteq 0 \wedge \bigwedge_{j \leq s} \bar{q}_j(b_i, \zeta) \doteq 0 \right) \text{ for } i < \omega.$$

Rewrite this as

$$\models \exists \zeta \in P^r \left( \neg \zeta \doteq 0 \wedge \bigwedge_{j \leq t} \bar{\mathcal{M}}(m, b_i) \cdot \bar{s}_j(e, \zeta) \doteq 0 \right) \text{ for } i < \omega$$

for given tuples  $e$  in  $E$  and  $m$  in  $M$ , a certain enumeration  $\bar{\mathcal{M}}(X, Y)$  of all monomials appearing in the the polynomials  $\bar{q}_j$ 's and tuples of polynomials  $\bar{s}_j(U, Z)$ 's with integer coefficients and homogeneous in  $\zeta$ .

Observe that the sequence  $(\mathcal{M}(m, b_i))_{i \leq \omega}$  is indiscernible over  $\emptyset$ . Thus, it suffices to show that

$$\varphi_1(x, y; e) = \exists \zeta \in P^r \left( \neg \zeta \doteq 0 \wedge \bigwedge_{j \leq t} \bar{\mathcal{M}}(x, y) \cdot \bar{s}_j(e, \zeta) \doteq 0 \right)$$

is indiscernibly closed. Lemma 4.1 yields the desired result, since the formula  $\varphi_1(x, y; u) \wedge u \in P$  is an equation.  $\square$

Corollary 3.5 and Proposition 4.2 yield the desired result:

**Theorem 4.3.** *The theory of proper pairs of algebraically closed fields of a fixed characteristic is equational.*

## 5. AN ALTERNATIVE PROOF IN CHARACTERISTIC 0

In this section, we will provide an alternative proof to the equationality of the theory  $T_P$  of belles paires of algebraically closed fields in characteristic 0, by means of differential algebra, based on an idea of Günaydın [5].

A differential field consists of a field  $K$  together with a distinguished additive morphism  $\delta$  satisfying Leibniz rule

$$\delta(xy) = x\delta(y) + y\delta(x).$$

Analogously to Zariski-closed sets for pure field, one defines *Kolchin-closed* sets in differential fields as zero sets of systems of differential-polynomial equations, that is, polynomial equations on the different iterates of the variables under the derivation. For a tuple  $x = (x_1, \dots, x_n)$  in  $K$ , denote by  $\delta(x)$  the tuple  $(\delta(x_1), \dots, \delta(x_n))$ . Ritt-Raudenbush's theorem implies that the Kolchin topology in differential fields of characteristic zero is Noetherian. Noetherianity holds in arbitrary characteristic and is easily proved:

**Lemma 5.1.** *In any differential field  $(K, \delta)$ , an algebraic differential equation*

$$p(x, \delta x, \dots; y, \delta y, \dots) \doteq 0$$

*is an equation in the model-theoretic sense.*

*Proof.* Write  $p(x, \delta x, \dots; a, \delta a, \dots)$  as  $q(M_1(x), \dots, M_m(x))$ , where  $q(u)$  is a linear polynomial with coefficients in  $K$  and the  $M_i$  are differential monomials in  $x$ . The result now follows from the trivial fact that linear equations  $q(u) \doteq 0$  are Srou closed in the model-theoretic sense and differential monomial are 0-definable in  $(K, \delta)$ .  $\square$

Perfect fields of positive characteristic cannot have non-trivial derivations. In characteristic zero though, any field  $K$  which is not algebraic over the prime field has a non-trivial derivation  $\delta$ . Furthermore, if  $K$  is algebraically closed, the set of *constants*  $E = \{a \in K \mid \delta(a) = 0\}$  is a proper algebraically closed subfield, so  $(K, E)$  is a model of  $T_P$ . We work from now on in a saturated algebraically closed differential field  $(K, \delta)$ , with non-zero  $\delta$ . For example in a saturated model of DCF<sub>0</sub>, the elementary theory of differential closed fields of characteristic zero.

In order to show that the theory of proper extensions of algebraically closed fields in characteristic 0, it suffices to show, by Proposition 3.4, that every instance of a tame formula determines a Kolchin-closed set in  $(K, \delta)$ . We first need a couple of auxiliary lemmata to describe the differential ideal associated to a system of polynomial equations.

**Lemma 5.2.** *Let  $v$  be a vector in  $K^n$ . Then the  $K$ -vector space generated by  $v, \delta(v), \dots, \delta^{n-1}(v)$  has a basis in  $E^n$ .*

*Proof.* Let  $V$  denote the span of  $v, \delta(v), \dots, \delta^{n-1}(v)$  and  $k$  be minimal such that  $v$  can be written as

$$v = a_1 e_1 + \dots + a_k e_k$$

for  $a_i \in K$  and  $e_i \in E^n$ . Clearly the  $e_i$  are linearly independent and  $k \leq n$ . We claim that  $e_1, \dots, e_k$  is a basis of  $V$ . Since

$$\delta^j(v) = \delta(a_1)e_1 + \dots + \delta(a_k)e_k,$$



the vector space  $V$  is contained in the span of the  $e_i$ . If the dimension of  $V$  is smaller than  $k$ , then  $v, \delta(v), \dots, \delta^{k-1}(v)$  are  $K$ -linearly dependent, that is the rows of the *Wronskian*

$$\begin{pmatrix} a_1 & a_2 & \dots & a_k \\ \delta(a_1) & \delta(a_2) & \dots & \delta(a_k) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{k-1}(a_1) & \delta^{k-1}(a_2) & \dots & \delta^{k-1}(a_k) \end{pmatrix}$$

are linearly dependent over  $K$ . It follows that  $a_1, \dots, a_k$  are linearly dependent over  $E$ . So there are  $\xi_i \in E$ , not all zero, such that  $\xi_1 a_1 + \dots + \xi_k a_k = 0$ . Then the vector space

$$\left\{ \sum_{i=1}^k b_i e_i \mid \sum_{i=1}^k \xi_i b_i = 0 \right\}$$

contains  $v$ , has a basis from  $E^n$  and its dimension is strictly smaller than  $k$ . This contradicts the choice of the  $e_i$ .  $\square$

**Corollary 5.3.** *Let  $V \subset K^n$  be the smallest subspace of  $K^n$  containing the vectors  $v_0, \dots, v_{m-1}$  and closed under  $\delta$ . Then  $V$  is generated by  $\{\delta^j(v_i)\}_{\substack{i < m \\ j < n}}$  and has a basis from  $E^n$ .*

Every polynomial over  $K[X]$  is a  $K$ -linear combination of monomials. Furthermore, homogeneous polynomials of a given degree are  $K$ -linear combinations of the corresponding monomials of the same degree. Thus, in order to apply the previous result, equip the polynomial ring  $K[X]$  in  $n$  variables with a derivation  $D$  obtained by differentiating the coefficients of a polynomial in  $K$ . Elements in  $E[X]$  have  $D$ -derivation 0. We say that an ideal  $I$  of  $K[X]$  is *differential* if it is closed under  $D$ . Clearly, any ideal generated by polynomials from  $E[X]$  is differential. Given a (finite) collection of polynomials  $\{h_i\}_{i \in I}$  of  $K[X]$ , the ideal generated by  $\{D^j(h_i)\}_{\substack{i \in I \\ j \in \mathbb{N}}}$  is finitely generated and a differential ideal. An easy application of Corollary 5.3 provides an upper bound for the set of generators of the latter.

**Corollary 5.4.** *An ideal of  $K[X]$  is differential if and only if it can be generated by elements from  $E[X]$ .*

Analogously, we conclude the corresponding result for homogeneous ideals.

**Corollary 5.5.** *Given homogeneous polynomials  $h_0, \dots, h_{m-1}$  in  $K[X]$  of a fixed degree  $d$ , there exists an integer  $k$  in  $\mathbb{N}$  (bounded only in terms of  $d$  and the length of  $X$ ) such that the ideal generated by  $\{D^j(h_i)\}_{\substack{i < m \\ j < k}}$  is a differential and homogeneous ideal.*

We now have all the ingredients in order to show that tame formulae are equations.

**Proposition 5.6.** *Every tame formula in  $(K, E)$  with parameters describes a Kolchin-closed set in  $(K, \delta)$ .*

*Proof.* Suppose that the tame formula  $\varphi(x)$  is of the form

$$\varphi(x) = \exists \zeta \in P^r \left( \neg \zeta \doteq 0 \wedge \bigwedge_{i < m} p_i(x, \zeta) \doteq 0 \right),$$

for polynomials  $p_0(X, Z), \dots, p_{m-1}(X, Z)$  over  $K$  homogeneous in  $Z$  of some fixed degree  $d$ . Let  $k$  be as in Corollary 5.5.

For a tuple  $b$  in  $K$  of length  $|x|$ , write

$$D^j(p_i(b, Z)) = p_{i,j}(b, \dots, \delta^j(b), Z),$$

for polynomials  $p_{i,j}(X_0, \dots, X_k, Z)$  over  $K$ , homogeneous in  $Z$ . By Corollary 5.5, the ideal  $I_b(Z)$  generated by

$$\{p_{i,j}(b, \dots, \delta^j(b), Z)\}_{\substack{i < m \\ j < k}}$$

has a generating set consisting of homogenous polynomials

$$g_0(Z), \dots, g_{s-1}(Z)$$

with coefficients in  $E[Z]$ .

Now, since  $\zeta$  ranges over the constant field, the tuple  $b$  realises  $\varphi(x)$  if and only if

$$(K, E) \models \exists \zeta \in P^r \left( \neg \zeta \doteq 0 \wedge I_b(\zeta) \doteq 0 \right),$$

which is equivalent to

$$(K, E) \models \exists \zeta \in P^r \left( \neg \zeta \doteq 0 \wedge \bigwedge_{i < s} g_i(\zeta) \doteq 0 \right),$$

Since the field  $E$  is an elementary substructure of  $K$ , this equivalent to

$$K \models \exists \zeta \left( \neg \zeta \doteq 0 \wedge \bigwedge_{i < s} g_i(\zeta) \doteq 0 \right),$$

which is again equivalent to

$$K \models \exists \zeta \left( \neg \zeta \doteq 0 \wedge I_b(\zeta) \doteq 0 \right).$$

Since  $I_b$  is homogeneous, the Zariski-closed set it determines is complete, hence its projection is given by a finite number of equations  $X(b, \dots, \delta^{k-1}(b))$ . Thus, the formula  $\varphi(b)$  holds if and only if

$$(K, \delta) \models X(b, \dots, \delta^{k-1}(b)),$$

which clearly describe a Kolchin-closed set, as desired.  $\square$

By Corollary 3.5, we conclude the following:

**Corollary 5.7.** *The theory of proper pairs of algebraically closed fields in characteristic 0 is equational.*

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